Limit theorems for Markov walks conditioned to stay positive

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1. Introduction
   - Formulation of the problem
   - Previous results : the independent case
   - General approach based on the coupling method in the independent case

2. Conditioned affine random walks
   - Problem A : the walk is not a martingale
   - Problem B : the killed martingale is not a submartingale
   - Problem C : the use of the Markov property at the time ν_n
   - Results

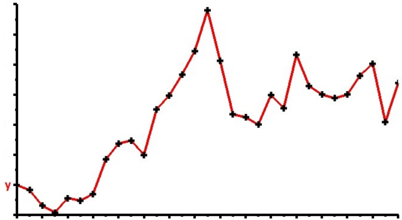
Limit theorems for Markov walks conditioned to stay positive

Let

- $y \in \mathbb{R}$ be a starting point,
- $(X_n)_{n \geqslant 1}$ be a sequence of real random variables.

We construct the random walk as follows :

$$\forall n \geqslant 1, \qquad y + S_n = y + X_1 + \cdots + X_n.$$



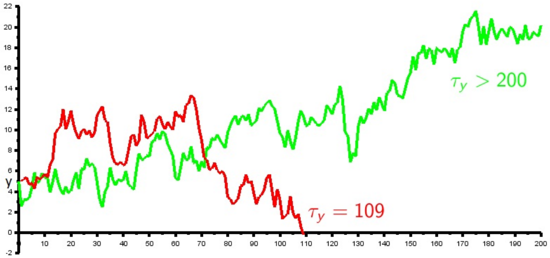
Limit theorems for Markov walks conditioned to stay positive

We define the first time for which the walk enter in the non-positive half-line :

$$\tau_y = \inf \{k \geqslant 1,\ y + S_k \leqslant 0\} .$$

The fact that the walk stays positive until the time n can be written as

$$\{y + S_1 > 0,\ y + S_2 > 0,\ \dots,\ y + S_n > 0\} = \{\tau_y > n\} .$$



Previous results : the independent case

We assume that the sequence $(X_n)_{n\geqslant 1}$ is

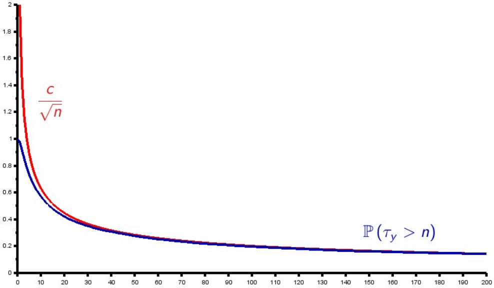
- independent, identically distributed,
- centred $\mathbb{E}\left(X_1\right) = 0$,
- with a moment of order 2, $\sigma^2 = \mathbb{E}\left(X_1^2\right) \in (0; +\infty)$.

Limit theorems for ~~Markov~~ walks conditioned to stay positive, the independent case (1/3)

Theorem (Spitzer, 1960)

Let $(X_n)_{n \geqslant 1}$ be i.i.d. If $\mathbb{E}(X_1) = 0$ and $0 < \mathbb{E}\left(X_1^2\right) < +\infty$, then for any $y \geqslant 0$, there exists $V(y) > 0$ such that

$$\mathbb{P}(\tau_y > n) \underset{n \to +\infty}{\sim} \frac{2V(y)}{\sqrt{2\pi n}\sigma}.$$



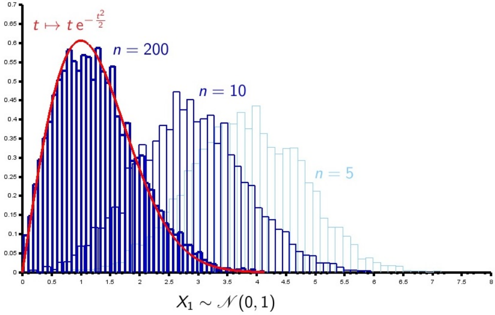
$X_1 \sim \mathcal{N}(0,1)$

Limit theorems for ~~Markov~~ walks conditioned to stay positive, the independent case (2/3)

Theorem (Iglehart, Bolthausen, 1974-1976)

Let $(X_n)_{n\geqslant 1}$ be i.i.d. If $\mathbb{E}(X_1) = 0$ and $0 < \mathbb{E}\left(X_1^2\right) < +\infty$, then for any $y > 0$ and $t \geqslant 0$,

$$\lim_{n\to+\infty} \mathbb{P}\left(y + S_n \leqslant t\sigma\sqrt{n} \,\middle|\, \tau_y > n \right) = 1 - e^{-\frac{t^2}{2}} .$$

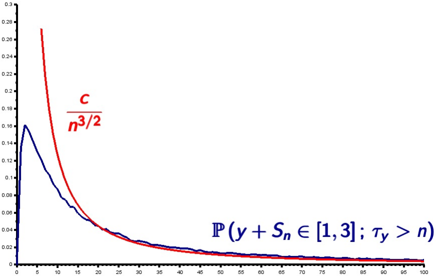


Limit theorems for ~~Markov~~ walks conditioned to stay positive, the independent case (3/3)

Conditioned local limit theorem (Iglehart, 1974, Vatutin, Wachtel, 2008)

Let $(X_n)_{n\geqslant 1}$ be i.i.d. If $\mathbb{E}(X_1) = 0$, $0 < \mathbb{E}\left(X_1^2\right) < +\infty$ and X_1 is non-lattice, then for any $y > 0$ and $0 \leqslant a < b$,

$$\mathbb{P}\left(y + S_n \in [a, b]\,;\, \tau_y > n\right) \underset{n\to+\infty}{\sim} \frac{\int_a^b V(z)\,\mathrm{d}z}{\sqrt{2\pi}\, n^{3/2}\sigma}.$$



Limit theorems for Markov walks conditioned to stay positive

Objective

To generalise the results when the increments $(X_n)_{n\geqslant 1}$ are a Markov chain (and not necessarily independent).

$$\mathbb{P}\left(\tau_y > n\right) \underset{n\to+\infty}{\sim} ?$$

$$\mathbb{P}\left(y + S_n \leqslant t\sqrt{n} \,\middle|\, \tau_y > n\right) \underset{n\to+\infty}{\sim} ?$$

$$\lim_{n\to+\infty} \mathbb{P}\left(y + S_n \in [a,b] \,\middle|\, \tau_y > n\right) ?$$

E. Presman
(1967) Boundary problems for sums of lattice random variables, defined on a finite regular markov chain. *Theory of Probability and Its Applications*.
(1969) Methods of factorization and a boundary problems for sums of random variables defined on a markov chain. *Izvestija Akademii Nauk SSSR*.

General approach based on the coupling method

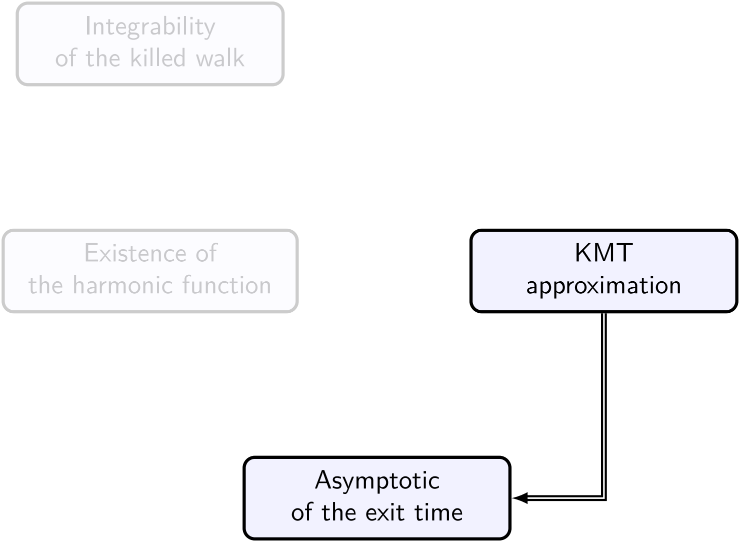
Integrability
of the killed walk

Existence of
the harmonic function

KMT
approximation

Asymptotic
of the exit time

General approach based on the coupling method



The coupling

Theorem [Komlós-Major-Tusnády]

Let $(X_n)_{n\geqslant 1}$ i.i.d., $\mathbb{E}(X_1) = 0$, $0 < \mathbb{E}(|X_1|^\alpha) < +\infty$, $\alpha > 2$. There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, without loss of generality one can reconstruct the sequence $(S_n)_{n\geqslant 1}$ together with a continuous time Brownian motion $(B_t)_{t\geqslant 0}$ such that for any $n \geqslant 1$,

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} \left|S_{\lfloor nt \rfloor} - \sigma B_{tn}\right| > n^{1/2-\epsilon}\right) \leqslant \frac{c_\epsilon}{n^\epsilon}.$$

$$\tau_y^{bm} = \inf\{t \geqslant 0 : y + \sigma B_t \leqslant 0\}.$$

We have

$$\mathbb{P}(\tau_y > n) \leqslant \mathbb{P}\left(\tau^{bm}_{y+n^{1/2-2\epsilon}} > n\right) + \frac{c_\epsilon}{n^{2\epsilon}}.$$

If $y \geqslant n^{1/2-\epsilon}$, we obtain that

$$\mathbb{P}(\tau_y > n) \leqslant \mathbb{P}\left(\tau^{bm}_{y\left(1+\frac{1}{n^\epsilon}\right)} > n\right) + \frac{c_\epsilon}{n^{2\epsilon}} \leqslant \frac{2y}{\sqrt{2\pi n}\sigma}\left(1 + \frac{c_\epsilon}{n^\epsilon}\right).$$

The stopping time ν_n

For any $y \in \mathbb{R}$, we define

$$\nu_n := \min\left\{k \geqslant 1 : y + S_k > n^{1/2-\epsilon}\right\}.$$

By the Markov property, for any $n \geqslant 1$,

$$\begin{aligned}
\mathbb{P}(\tau_y > n) &= \mathbb{P}\left(\tau_y > n\,,\, \nu_n \leqslant \left\lfloor n^{1-\epsilon}\right\rfloor\right) + \mathbb{P}\left(\tau_y > n\,,\, \nu_n > \left\lfloor n^{1-\epsilon}\right\rfloor\right)\\
&= \sum_{k=1}^{\lfloor n^{1-\epsilon}\rfloor} \int_0^{+\infty} \mathbb{P}(\tau_{y'} > n-k)\,\mathbb{P}(y+S_k \in \mathrm{d}y'\,,\, \tau_y > k\,,\, \nu_n = k) + R_n\\
&\leqslant \sum_{k=1}^{\lfloor n^{1-\epsilon}\rfloor} \int_0^{+\infty} \frac{2y'}{\sqrt{2\pi n}\sigma}\left(1+\frac{c_\epsilon}{n^\epsilon}\right)\mathbb{P}(y+S_k \in \mathrm{d}y'\,,\, \tau_y > k\,,\, \nu_n = k) + R_n\\
&= \frac{2}{\sqrt{2\pi n}\sigma}\left(1+\frac{c_\epsilon}{n^\epsilon}\right)\mathbb{E}\left(y+S_{\nu_n}\,;\, \tau_y > \nu_n\,,\, \nu_n \leqslant \left\lfloor n^{1-\epsilon}\right\rfloor\right) + R_n.
\end{aligned}$$

The function V

Definition/Proposition 1.1

The following limit exists and is denoted by

$$V(y) := \lim_{n \to +\infty} \mathbb{E}\left(y + S_n\,;\, \tau_y > n\right) \in (0, +\infty).$$

Lemma 1.2

For any $y \in \mathbb{R}$,

$$\lim_{n \to +\infty} \mathbb{E}\left(y + S_{\nu_n}\,;\, \tau_y > \nu_n\,,\, \nu_n \leqslant \left\lfloor n^{1-\epsilon} \right\rfloor\right) = V(y).$$

$$\mathbb{P}\left(\tau_y > n\right) = \frac{2}{\sqrt{2\pi n}\sigma}\left(1 + \frac{c_\epsilon}{n^\epsilon}\right)\mathbb{E}\left(y + S_{\nu_n}\,;\, \tau_y > \nu_n\,,\, \nu_n \leqslant p\right) + R_n.$$

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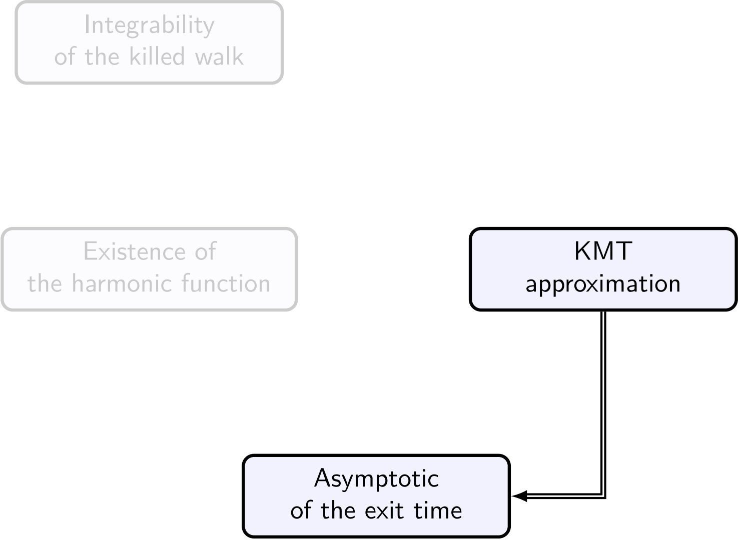
Theorem 1.3

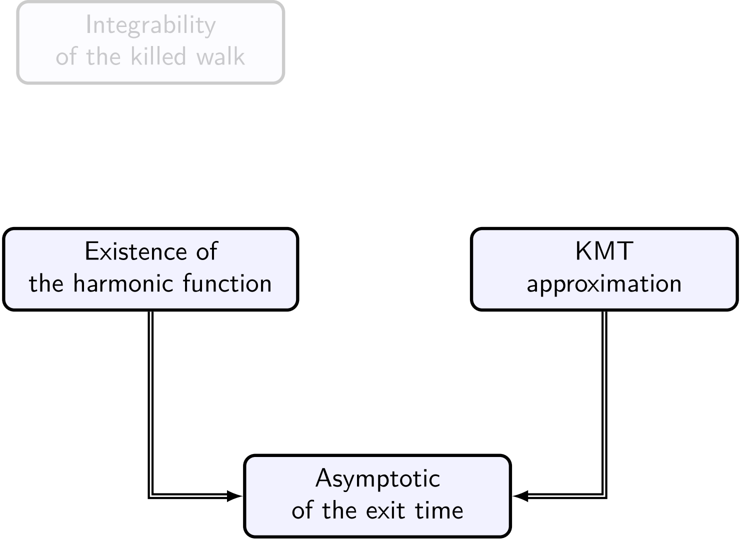
For any $y \in \mathbb{R}$,

$$\mathbb{P}\left(\tau_y > n\right) \underset{n \to +\infty}{\sim} \frac{2V(y)}{\sqrt{2\pi n}\sigma}.$$

General approach based on the coupling method



General approach based on the coupling method



The harmonic function

Definition/Proposition 1.1

The following limit exists and is denoted by

$$V(y) := \lim_{n\to+\infty} \mathbb{E}\left(y + S_n\,;\, \tau_y > n\right) \in (0, +\infty).$$

Proposition 1.4

The function V is harmonic : for any $y \in \mathbb{R}$,

$$V(y) = \mathbb{E}\left(V\left(y + S_1\right)\,;\, \tau_y > 1\right).$$

Existence of the harmonic function

$$V(y) = \lim_{n \to +\infty} V_n(y) = \lim_{n \to +\infty} \mathbb{E}\left(y + S_n \,;\, \tau_y > n\right) \in (0, +\infty).$$

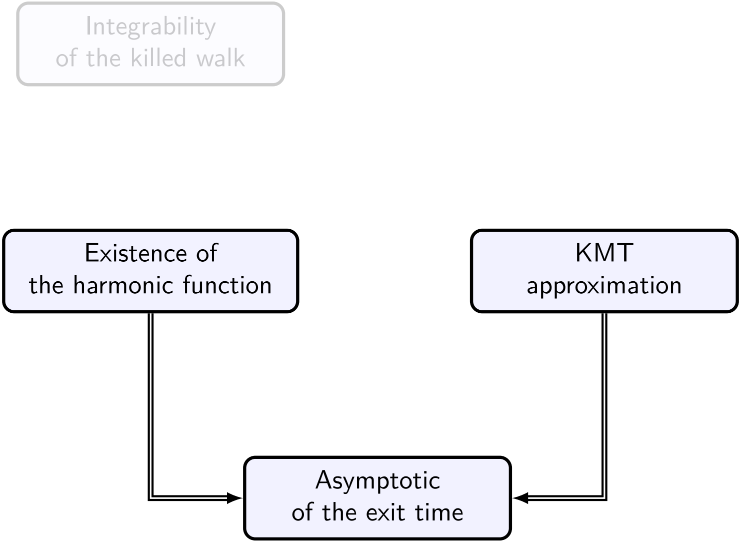
Lemma 1.5

The killed random walk $\left((y + S_n)\mathbb{1}_{\{\tau_y > n\}}\right)_{n \geqslant 0}$ is a submartingale :

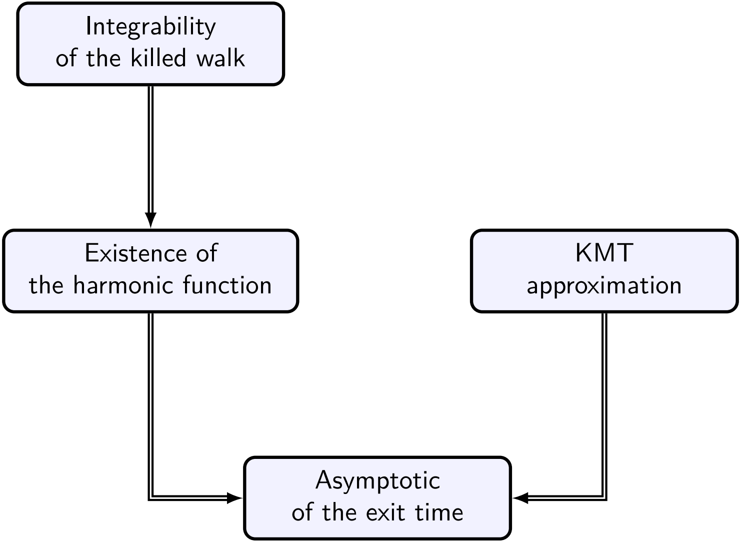
$$V_{n+1}(y) := \mathbb{E}\left(y + S_{n+1} \,;\, \tau_y > n+1\right) \geqslant \mathbb{E}\left(y + S_n \,;\, \tau_y > n\right) =: V_n(y).$$

$\Rightarrow$ It remains to prove that the sequence $\left(V_n(y)\right)_{n \geqslant 0}$ is bounded.

General approach based on the coupling method



General approach based on the coupling method



Integrability of the killed random walk (1/2)

Lemma 1.6

There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, $y \in \mathbb{R}$ and $n \geqslant 1$,

$$\mathbb{E}\left(y + S_n\,;\, \tau_y > n\right) \leqslant y + c_\epsilon n^{1/2-2\epsilon}.$$

The proof is based on the fact that

$$0 \geqslant y + S_{\tau_y} = y + S_{\tau_y - 1} + X_{\tau_y} > X_{\tau_y}.$$

Integrability of the killed random walk (1/2)

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The proof is based on the fact that

$$0 \geqslant y + S_{\tau_y} = y + S_{\tau_y - 1} + X_{\tau_y} > X_{\tau_y}.$$

If $y \geqslant n^{1/2-\epsilon}$,

$$\mathbb{E}\left(y + S_n \,;\, \tau_y > n\right) \leqslant \left(1 + \frac{c_\epsilon}{n^\epsilon}\right) y.$$

For any $y \in \mathbb{R}$, we define

$$\nu_n := \min\left\{k \geqslant 1 : y + S_k > n^{1/2-\epsilon}\right\}.$$

Integrability of the killed random walk (2/2)

Lemma 1.8

For any $y \in \mathbb{R}$ and $n \geqslant 1$,

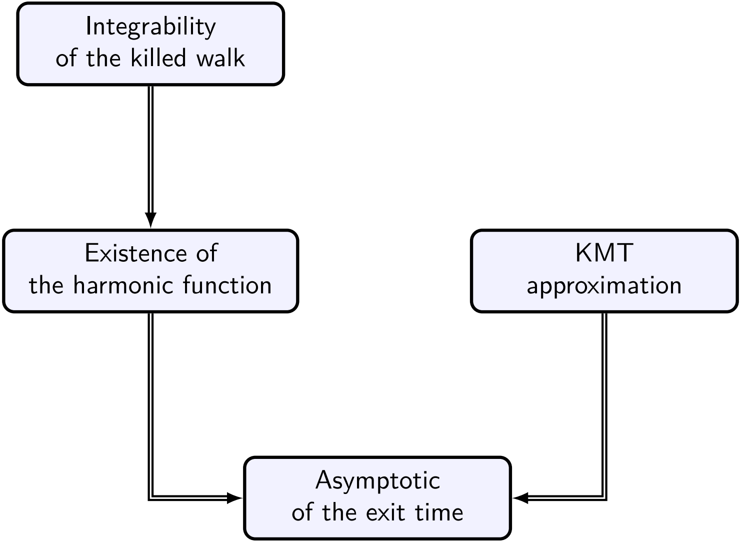
$$y \leqslant \mathbb{E}\left(y + S_n \,;\, \tau_y > n\right) \leqslant c(1+y).$$

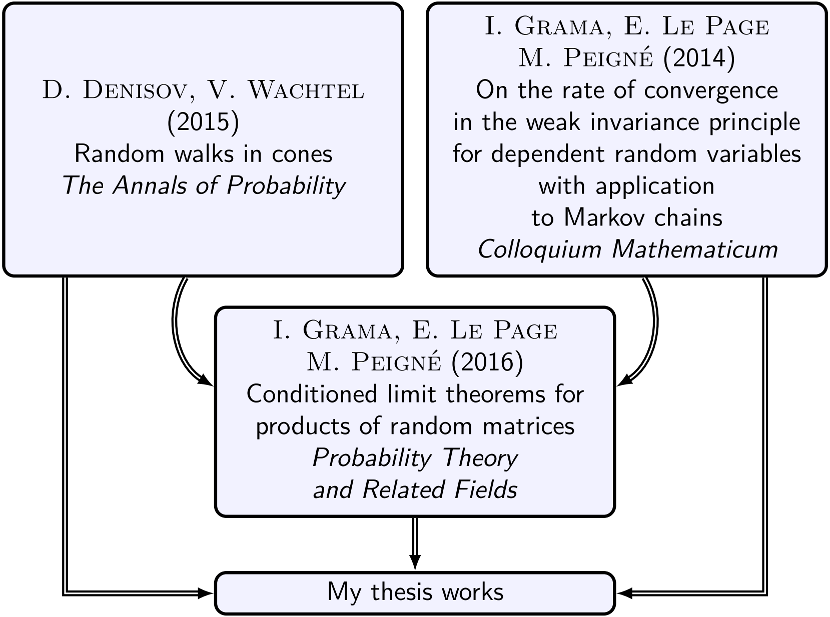
Definition/Proposition 1.1

The following limit exists and is denoted by

$$V(y) := \lim_{n \to +\infty} \mathbb{E}\left(y + S_n \,;\, \tau_y > n\right) \in (0, +\infty).$$

General approach based on the coupling method




D. Denisov, V. Wachtel
(2015)
Random walks in cones
The Annals of Probability
I. Grama, E. Le Page
M. Peigné (2014)
On the rate of convergence
in the weak invariance principle
for dependent random variables
with application
to Markov chains
Colloquium Mathematicum
I. Grama, E. Le Page
M. Peigné (2016)
Conditioned limit theorems for
products of random matrices
Probability Theory
and Related Fields
My thesis works

Year 1 : Conditioned affine Markov walks

Year 2 : Conditioned Markov walks with a spectral gap

Year 3 : Branching processes in Markov environment

Year 3 : Conditioned local limit theorems

The model

Let $(a_n, b_n)_{n \geqslant 1}$ a sequence of random variables i.i.d. The increments of the walk are defined as follows. For any $n \geqslant 0$,

$$X_{n+1} = a_{n+1} X_n + b_{n+1} \qquad \text{and} \qquad X_0 = x \in \mathbb{R}.$$

Condition 2.1

1. There exists $\alpha > 2$ such that $\mathbb{E}\left(|a_1|^{\alpha}\right) < 1$ and $\mathbb{E}\left(|b_1|^{\alpha}\right) < +\infty$.
2. The random variable b_1 is non-zero with positive probability, $\mathbb{P}\left(b_1 \neq 0\right) > 0$ and centred $\mathbb{E}\left(b_1\right) = 0$.

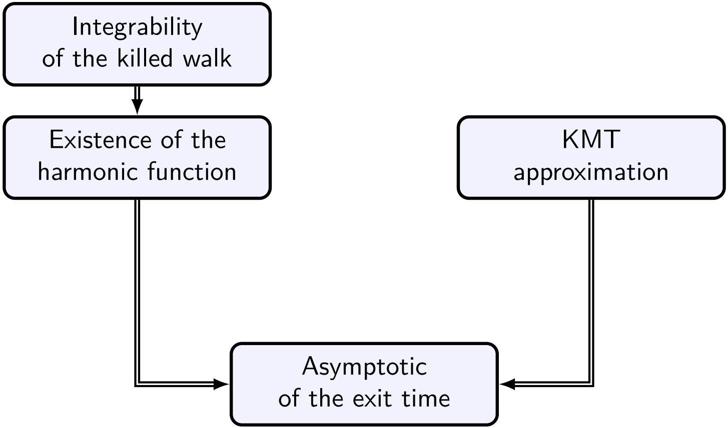
$\Rightarrow (X_n)_{n \geqslant 0}$ is a Markov chain with a unique invariant measure.

Y. Guivarc'h, E. Le Page (2008). On spectral properties of a family of transfer operators and convergence to stable laws for affine random walks. *Ergodic Theory and Dynamical Systems*.

For any $y \in \mathbb{R}$ and $n \geqslant 1$, the random walk is given by

$$y + S_n = y + \sum_{k=1}^{n} X_k.$$

Problem A


Integrability
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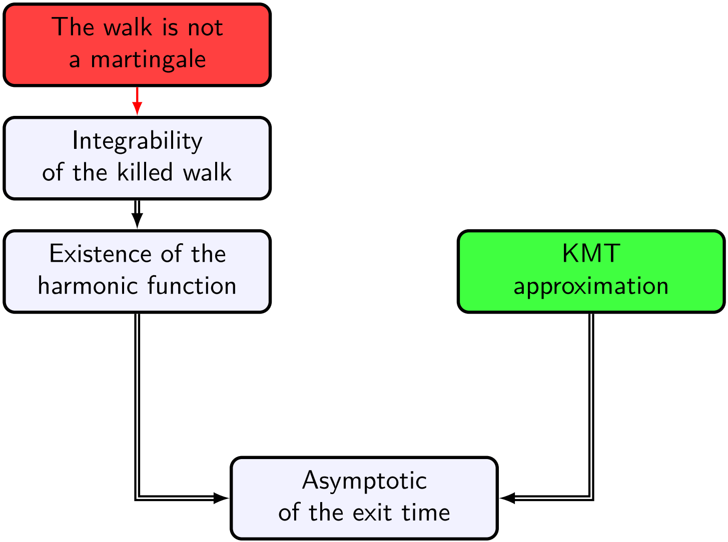
Problem A


The walk is not a martingale
Integrability of the killed walk
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Martingale approximation

For any $n \geqslant 1$, let

$$M_n = \sum_{k=1}^{n} \Theta\left(X_k\right) - \mathbf{P}\Theta\left(X_{k-1}\right),$$

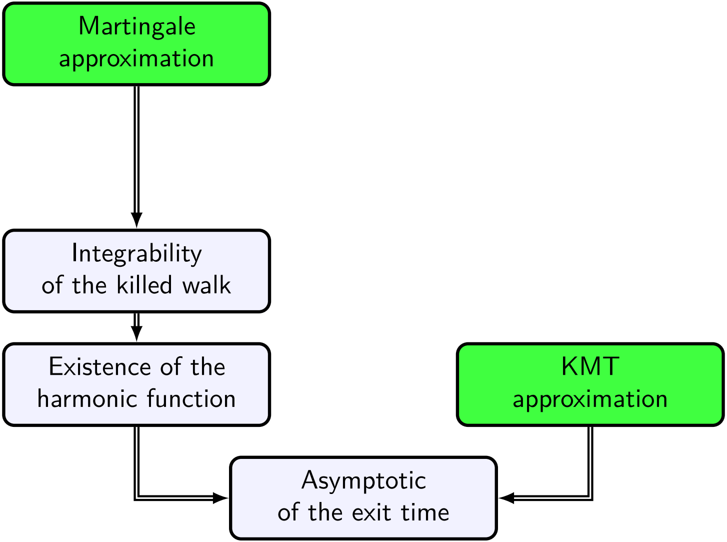
where Θ is the solution of the Poisson equation. Then, $(M_n)_{n \geqslant 1}$ is a martingale. Moreover,

$$y + S_n = z + M_n - \rho X_n,$$

where

$$\rho = \frac{\mathbb{E}(a)}{1 - \mathbb{E}(a)} \qquad \text{and} \qquad z = y + \rho x.$$

Problem B


Martingale approximation
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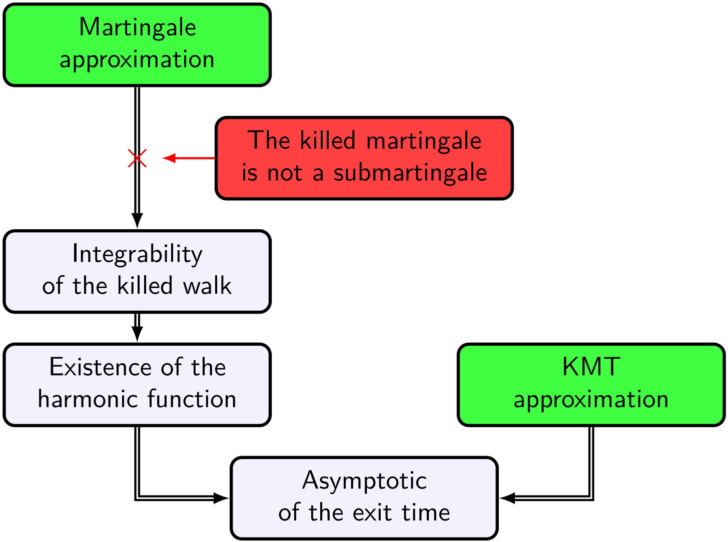
Problem B


Martingale approximation
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The killed martingale, case $\mathbb{E}(a) \geqslant 0$

Lemma 1.5

The killed random walk $\left((y+S_n)\mathbb{1}_{\{\tau_y>n\}}\right)_{n\geqslant 0}$ is a submartingale :

$$\mathbb{E}\left(y+S_{n+1}\,;\,\tau_y>n+1\right) \geqslant \mathbb{E}\left(y+S_n\,;\,\tau_y>n\right).$$

Lemma 2.4.2

Assume Condition 2.1 and $\mathbb{E}(a) \geqslant 0$.

1. For all $x \in \mathbb{R}$ and $y > 0$,

$$\frac{X_{\tau_y}}{1-\mathbb{E}(a)} < z + M_{\tau_y} \leqslant 0.$$

2. For all $x \in \mathbb{R}$ and $y > 0$, the sequence $\left((z+M_n)\mathbb{1}_{\{\tau_y>n\}}\right)_{n\geqslant 0}$ is a submartingale.

The killed martingale, case $\mathbb{E}(a) < 0$

Definition

For any $x \in \mathbb{R}$ and $y \in \mathbb{R}$,

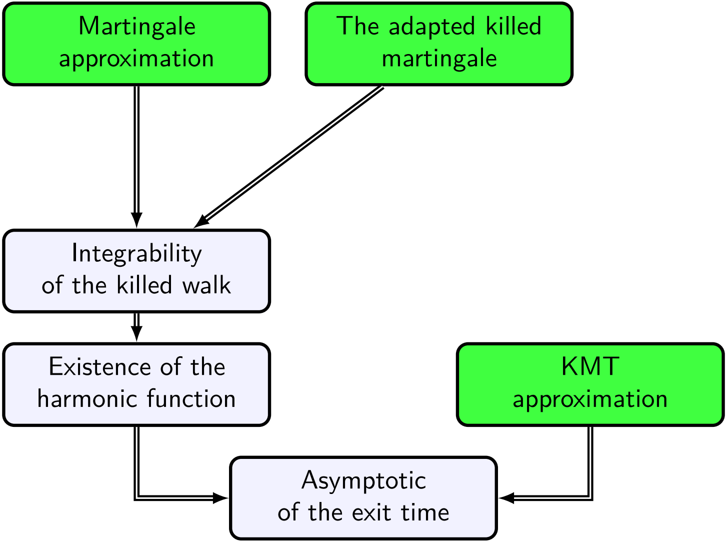
$$T_y = \min\left\{k \geqslant 1 : z + M_k \leqslant 0\right\}$$

Lemma 2.4.6

Assume Condition 2.1 and $\mathbb{E}(a) < 0$.

1. For all $x \in \mathbb{R}$ and $y > 0$,
$$\tau_y \leqslant T_y$$
2. For all $x \in \mathbb{R}$ and $y > 0$, the sequence $\left((z + M_n)\mathbb{1}_{\{T_y > n\}}\right)_{n \geqslant 0}$ is a submartingale.

Problem C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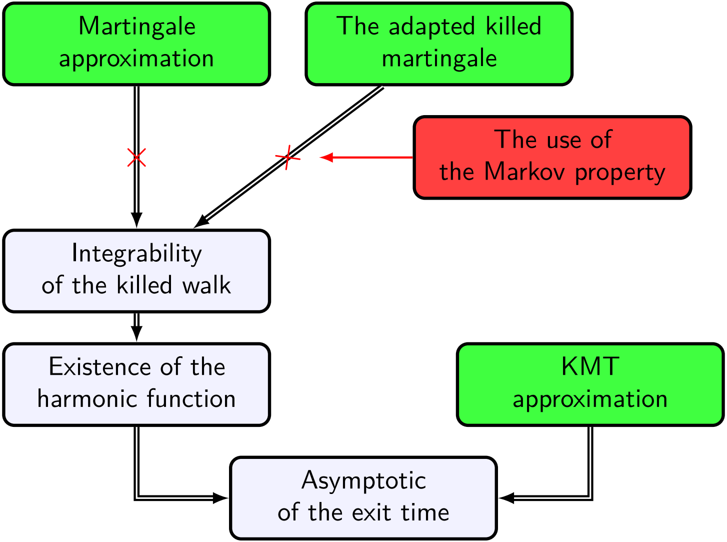
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The Markov property

Lemma 1.6

There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, $y \in \mathbb{R}$ and $n \geqslant 1$,

$$\mathbb{E}\left(y + S_n \,;\, \tau_y > n\right) \leqslant y + c_\epsilon n^{1/2-2\epsilon}.$$

$$\begin{aligned}&\mathbb{E}\left(y + S_n \,;\, \tau_y > n\right)\\ &\qquad \leqslant \left(1 + \frac{c_\epsilon}{n^\epsilon}\right) \mathbb{E}\left(y + S_{\nu_n} \,;\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\epsilon} \right\rfloor\right) + c_\epsilon \,\mathrm{e}^{-c_\epsilon n^\epsilon}\,.\end{aligned}$$

The Markov property

Lemma 2.4.3

There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, $y > 0$, $x \in \mathbb{R}$ and $n \geqslant 1$,

$$\mathbb{E}_x \left(z + M_n \,;\, \tau_y > n \right) \leqslant z + c_\epsilon n^{1/2-2\epsilon} + c \left| x \right|.$$

$$\begin{aligned} &\mathbb{E} \left(y + S_n \,;\, \tau_y > n \right) \\ &\qquad \leqslant \left(1 + \frac{c_\epsilon}{n^\epsilon} \right) \mathbb{E} \left(y + S_{\nu_n} \,;\, \tau_y > \nu_n \,,\, \nu_n \leqslant \left\lfloor n^{1-\epsilon} \right\rfloor \right) + c_\epsilon \, e^{-c_\epsilon n^\epsilon} . \end{aligned}$$

The Markov property

Lemma 2.4.3

There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, $y > 0$, $x \in \mathbb{R}$ and $n \geqslant 1$,

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$$\begin{aligned}
&\mathbb{E}_x\left(z + M_n\,;\, \tau_y > n\right) \\
&\leqslant \mathbb{E}_x\left(\left(\left(1 + \frac{c_\epsilon}{n^\epsilon}\right)\left(z + M_{\nu_n}\right) + c\left|X_{\nu_n}\right|\,;\, \tau_y > \nu_n\,,\, \nu_n \leqslant \left\lfloor n^{1-\epsilon}\right\rfloor\right)\right) + R_n.
\end{aligned}$$

The Markov property

Lemma 2.4.3

There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, $y > 0$, $x \in \mathbb{R}$ and $n \geqslant 1$,

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The perturbed time

$$\nu_n^\epsilon := \nu_n + \left\lfloor n^\epsilon \right\rfloor.$$

The Markov property

$$\mathbb{E}_x\left(z + M_n \,;\, \tau_y > n\right) \leqslant \mathbb{E}_x\left(\left(1 + \frac{c_\epsilon}{n^\epsilon}\right)(z + M_{\nu_n}) + c\left|X_{\nu_n}\right| \,;\, \tau_y > \nu_n\,,\, \nu_n \leqslant \left\lfloor n^{1-\epsilon}\right\rfloor\right) + R_n.$$

The perturbed time

$$\nu_n^\epsilon := \nu_n + \left\lfloor n^\epsilon \right\rfloor.$$

Lemma 2.3.1

$$\mathbb{E}_x\left(|X_n|\right) \leqslant c + \mathrm{e}^{-cn}|x|.$$

$$c\mathbb{E}_x\left(\left|X_{\nu_n^\epsilon}\right| \,;\, \tau_y > \nu_n^\epsilon\,,\, \nu_n^\epsilon \leqslant \left\lfloor n^{1-\epsilon}\right\rfloor\right) \leqslant c\mathbb{E}_x\left(1 + \mathrm{e}^{-c\lfloor n^\epsilon\rfloor}\left|X_{\nu_n}\right| \,;\, \tau_y > \nu_n\,,\, \nu_n \leqslant \left\lfloor n^{1-\epsilon}\right\rfloor\right).$$

Existence of the harmonic function

Proposition 2.5.2

Assume Condition 2.1 and $\mathbb{E}(a) \geqslant 0$. For any $x \in \mathbb{R}$ and $y > 0$,

$$V(x,y) = \lim_{n \to +\infty} \mathbb{E}_x \left(z + M_n \, ; \, \tau_y > n \right).$$

Moreover, for any $p \in (2, \alpha)$,

$$\max(0, z) \leqslant V(x,y) \leqslant c_p \left(1 + y + |x|\right) \left(1 + |x|\right)^{p-1}.$$

Proposition 2.4.8

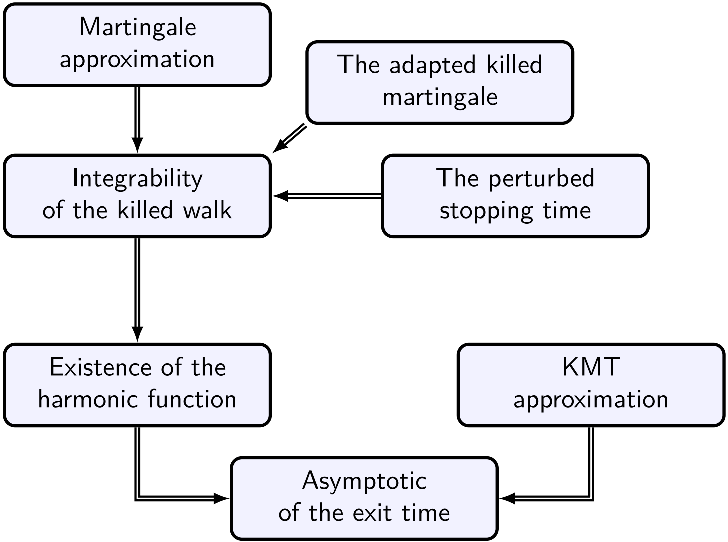
Assume Condition 2.1 and $\mathbb{E}(a) < 0$. For any $x \in \mathbb{R}$ and $y > 0$,

$$V(x,y) = \lim_{n \to +\infty} \mathbb{E}_x \left(z + M_n \, ; \, \tau_y > n \right).$$

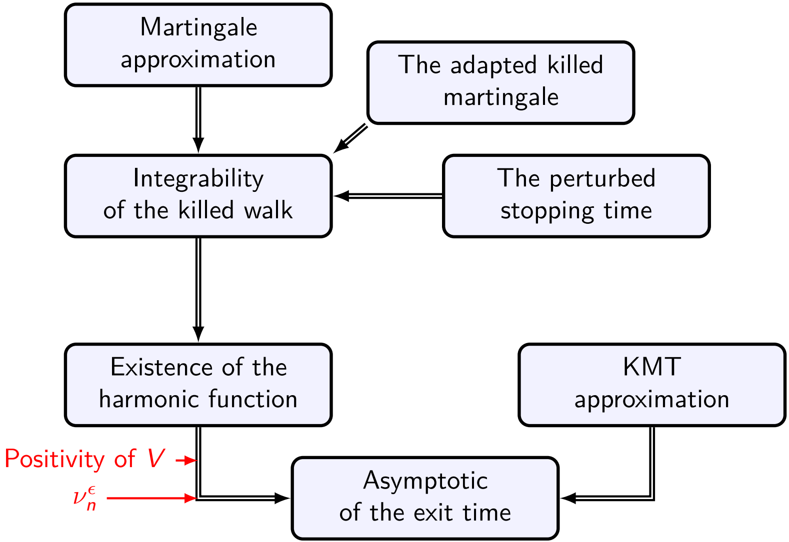
Moreover, for any $p \in (2, \alpha)$,

$$0 \leqslant V(x,y) \leqslant c_p \left(1 + y + |x|^p\right).$$

Conditioned affine random walks



Conditioned affine random walks



Conditioned affine random walks

Theorem 2.2.2

Assume Condition 2.1 and either Condition 2.2 and $\mathbb{E}(a) \geqslant 0$ or Condition 2.3, then for any $x \in \mathbb{X}$ and $y > 0$,

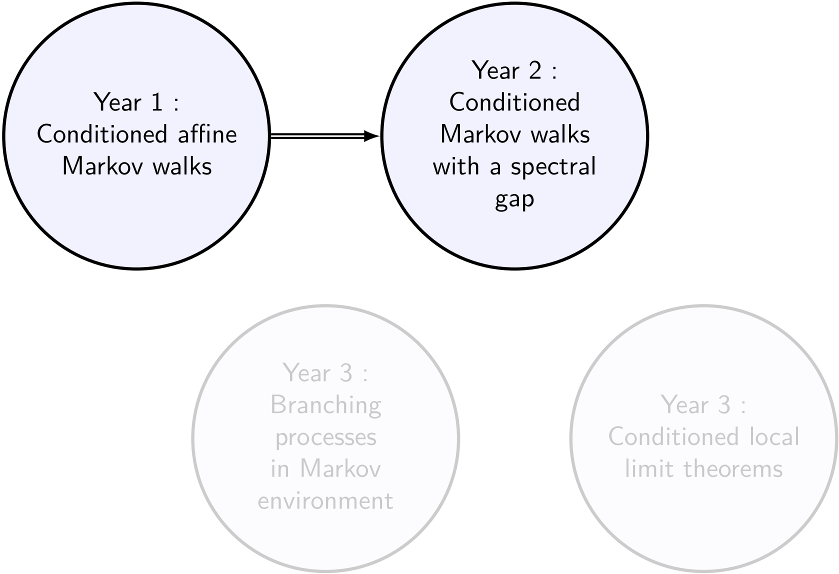
$$\mathbb{P}_x\left(\tau_y > n\right) \underset{n \to +\infty}{\sim} \frac{2V(x,y)}{\sqrt{2\pi n}\sigma}.$$

Theorem 2.2.4

Assume Condition 2.1 and either Condition 2.2 and $\mathbb{E}(a) \geqslant 0$ or Condition 2.3, then for any $x \in \mathbb{X}$, $y > 0$ and $t > 0$,

$$\lim_{n \to +\infty} \mathbb{P}_x\left(\frac{y + S_n}{\sigma\sqrt{n}} \leqslant t \,\middle|\, \tau_y > n \right) = \mathbf{\Phi}^+(t),$$

where $\mathbf{\Phi}^+(t) = 1 - e^{-\frac{t^2}{2}}$ is the Rayleigh distribution function.


Year 1 : Conditioned affine Markov walks
Year 2 : Conditioned Markov walks with a spectral gap
Year 3 : Branching processes in Markov environment
Year 3 : Conditioned local limit theorems

The spectral gap assumption

- Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain with values in an abstract state space $\mathbb{X}$.
- For a given real function $f : \mathbb{X} \to \mathbb{R}$, we set $S_n = \sum_{k=1}^{n} f(X_k)$.
- Let $\mathscr{B}$ be a Banach space of complex valued functions on $\mathbb{X}$.

Hypothesis M3.2

The transition operator $\mathbf{P}$ satisfies for any $g \in \mathscr{B}$:

$$\mathbf{P}g = \boldsymbol{\nu}(g)e + Qg,$$

where $\boldsymbol{\nu} \in \mathscr{B}'$ is a linear form,
e is the constant function on $\mathbb{X}$ equal to 1
and Q is an operator on $\mathscr{B}$ such that $Q(e) = 0$, $\boldsymbol{\nu} \circ Q = 0$ and for any $n \geqslant 1$,

$$\|Q^n\|_{\mathscr{B}\to\mathscr{B}} \leqslant c\,\mathrm{e}^{-cn}.$$

Examples of Markov walks with a spectral gap

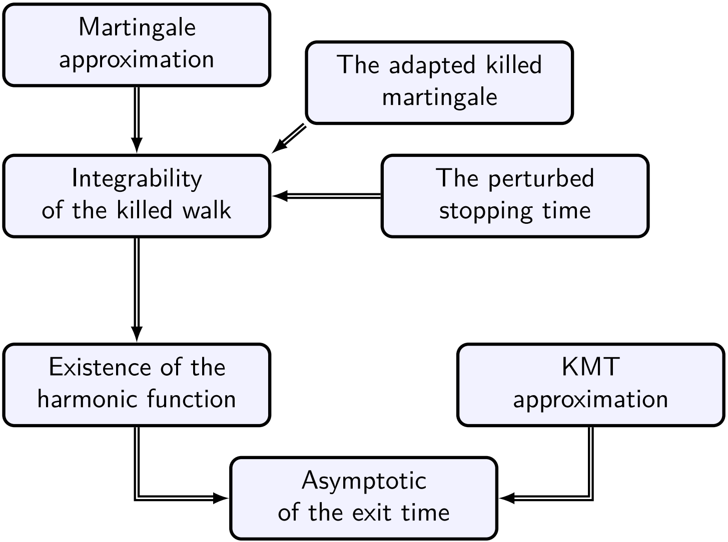
- The random walks with independent increments.
- The affine random walks.
- Markov walks with increments in a finite state space.
- Markov walks with increments satisfying the Doeblin-Fortet condition.
  Y. Guivarc'h and J. Hardy (1988). Théorèmes limites pour une classe de chaîne de Markov et applications aux difféomorphismes d'Anosov. *Annales de l'IHP Probabilités et statistiques*.
- Affine random walks in $\mathbb{R}^d$ conditioned to stay in a half-space :

$$X_{n+1} = A_{n+1}X_n + B_{n+1}, \qquad (A_{n+1}, B_{n+1}) \in GL(d, \mathbb{R}) \times \mathbb{R}^d$$

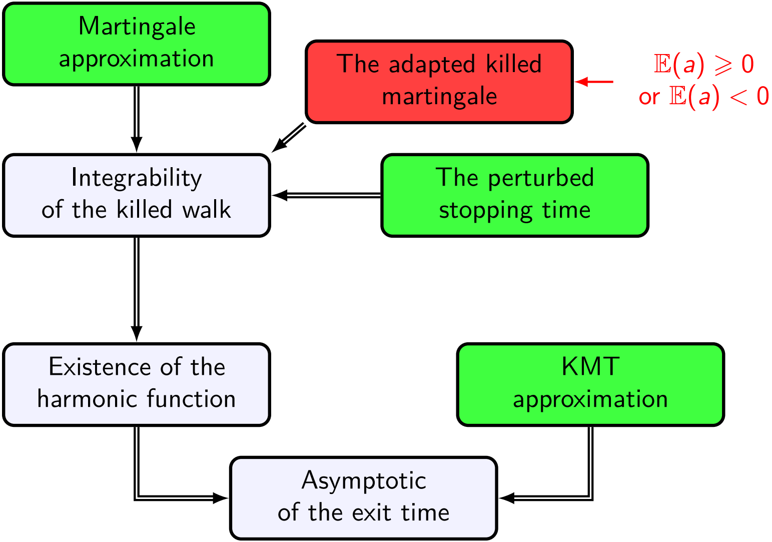
and for any $y \in \mathbb{R}$,

$$y + S_n = y + \sum_{k=1}^{n} \langle u, X_n \rangle.$$

How generalize the conditioned affine random walks



How generalize the conditioned affine random walks



The hybrid stopping time (1/2)

Definition

For any $y \in \mathbb{R}$ and $z \in \mathbb{R}$,

$$\tau_y := \min\{k \geqslant 1 : y + S_k \leqslant 0\}$$
$$T_z := \min\{k \geqslant 1 : z + M_k \leqslant 0\}.$$

Definition (The hybrid stopping time)

For any $x \in \mathbb{X}$ and $z \in \mathbb{R}$,

$$\hat{T}_z := \min\{k \geqslant \tau_y : z + M_k \leqslant 0\},$$

where $y = z - r(x)$ and $r(x) = \mathbf{P}\Theta(x)$.

The hybrid stopping time (2/2)

Definition

For any $x \in \mathbb{X}$ and $z \in \mathbb{R}$,

$$\hat{T}_z := \min\left\{k \geqslant \tau_y : z + M_k \leqslant 0\right\},$$

where $y = z - r(x)$ and $r(x) = \mathbf{P}\Theta(x)$.

Lemma 3.5.4

- For any $z \in \mathbb{R}$,
$$z + M_{\hat{T}_z} \leqslant 0$$
- For any $x \in \mathbb{X}$ and $y \in \mathbb{R}$,
$$\hat{T}_z \geqslant \max\left\{\tau_y, T_z\right\}$$
- The sequence $\left((z + M_n)\,\mathbb{1}_{\{\hat{T}_z > n\}}\right)_{n \in \mathbb{N}}$ is a submartingale.

$\longrightarrow$ The application of the Markov property for the hybrid time is different.

Integrability of the killed martingale

Lemma 2.4.4

Assume Condition 2.1 and $\mathbb{E}(a) \geqslant 0$. For any $p \in (2, \alpha)$, $x \in \mathbb{R}$, $y > 0$ and $n \geqslant 1$,

$$\mathbb{E}_x \left(z + M_n \, ; \, \tau_y > n \right) \leqslant c_p \left(1 + y + |x| \right) \left(1 + |x| \right)^{p-1}.$$

Lemma 2.4.8

Assume Condition 2.1. For any $p \in (2, \alpha)$, $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $n \geqslant 1$,

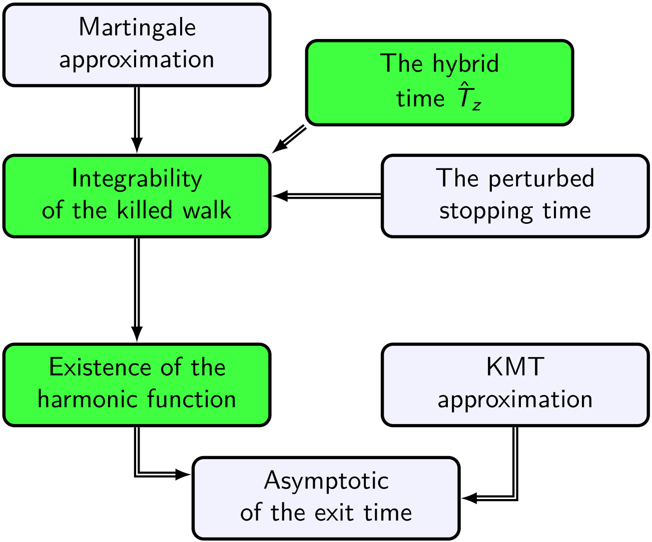
$$\mathbb{E}_x \left(z + M_n \, ; \, T_y > n \right) \leqslant c_p \left(1 + \max(y, 0) + |x|^p \right).$$

Lemma 3.6.4

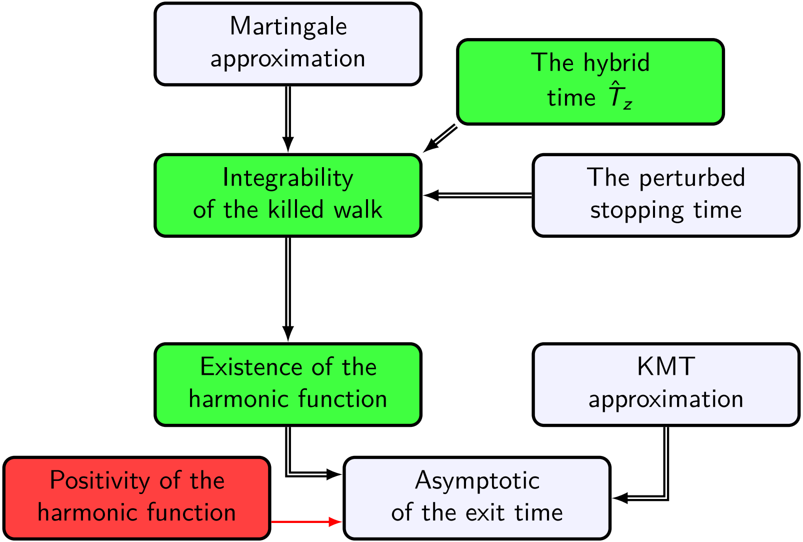
There exists $c > 0$ such that for any $x \in \mathbb{X}$, $z \in \mathbb{R}$ and $n \geqslant 1$,

$$\mathbb{E}_x \left(z + M_n \, ; \, \hat{T}_z > n \right) \leqslant c \left(1 + \max(z, 0) + N(x) \right).$$

Conditioned Markov walks with a spectral gap



Conditioned Markov walks with a spectral gap



Positivity of the harmonic function

Proposition 3.8.6

For any $\delta \in (0,1)$, $x \in \mathbb{X}$ and $y > 0$,

$$V(x,y) \geqslant (1-\delta)y - c_\delta \left(1 + N(x)\right).$$

D. Denisov and V. Wachtel (2010). Conditioned limit theorems for ordered random walks. *Electronic Journal of Probability*.

Definition

For any $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $\gamma > 0$,

$$\zeta_\gamma = \inf\left\{k \geqslant 1 : |y + S_k| > \gamma\left(1 + N\left(X_k\right)\right)\right\}$$
$$\mathscr{D}_\gamma = \left\{(x,y) \in \mathbb{X} \times \mathbb{R} : \exists n_0 \geqslant 1,\ \mathbb{P}_x\left(\zeta_\gamma \leqslant n_0, \tau_y > n_0\right) > 0\right\}.$$

Proposition 3.8.8

There exists $\gamma_0 > 0$ such that for any $\gamma \geqslant \gamma_0$,

$$\operatorname{supp}(V) = \mathscr{D}_\gamma.$$

Asymptotic of the exit time

Theorem 3.2.3

1. For any $(x,y) \in \operatorname{supp}(V)$,

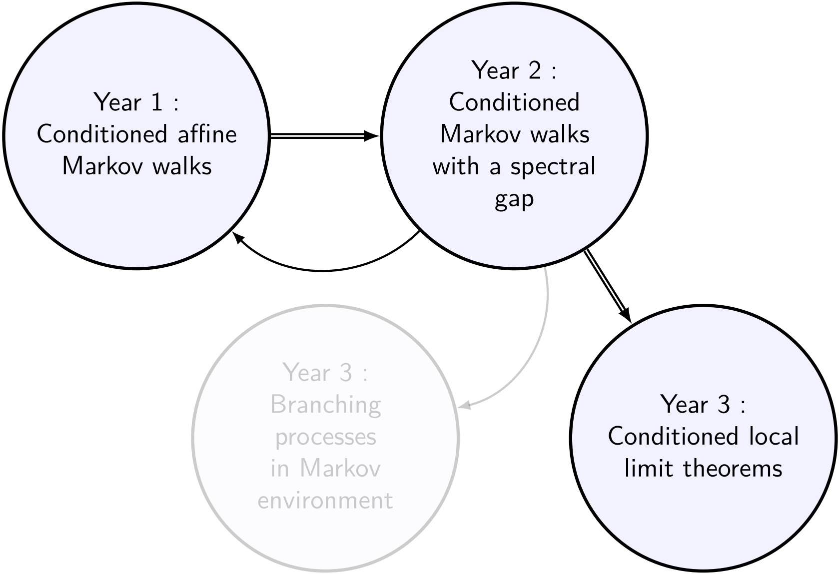
$$\mathbb{P}_x\left(\tau_y > n\right) \underset{n\to+\infty}{\sim} \frac{2V(x,y)}{\sqrt{2\pi n}\sigma}.$$

2. For any $(x,y) \notin \operatorname{supp}(V)$ and $n \geqslant 1$,

$$\mathbb{P}_x\left(\tau_y > n\right) \leqslant c\,\mathrm{e}^{-cn}\left(1+N(x)\right).$$

NB : In the i.i.d. case, we have for any $y \notin \operatorname{supp}(V)$ and $n \geqslant 1$,

$$\mathbb{P}\left(\tau_y > n\right) = 0.$$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Conditioned local limit theorems

Conditioned local limit theorems (CLLT)

- What is the survival probability of the walk until the time n ?

$$\mathbb{P}_x\left(\tau_y > n\right) \underset{n\to+\infty}{\sim} \frac{2V(x,y)}{\sqrt{2\pi n}\sigma}.$$

- What is the behaviour of the walk conditioned to stay positive ?

$$\text{Pour } t > 0, \qquad \lim_{n\to+\infty} \mathbb{P}_x\left(y + S_n \leqslant t\sqrt{n} \,\middle|\, \tau_y > n \right) = 1 - \mathrm{e}^{-\frac{t^2}{2}}.$$

- What it the *local* behaviour of the walk conditioned to stay positive ?

$$\text{Pour } 0 \leqslant a < b, \qquad \mathbb{P}_x\left(y + S_n \in [a,b] \,|\, \tau_y > n\right) \underset{n\to+\infty}{\sim} ?$$

Sketch of the proof of CLLT in i.i.d. case (1/3)

Theorem (Stone, 1965)

Let $(X_n)_{n \geqslant 1}$ be i.i.d. If $\mathbb{E}(X_1) = 0$, $0 < \mathbb{E}(X_1^2) < +\infty$ and X_1 is non-lattice, then for any $y \in \mathbb{R}$, $z \geqslant 0$ and $a > 0$,

$$\mathbb{P}(y + S_n \in [z, z+a]) \underset{n \to +\infty}{\sim} \frac{a}{\sqrt{2\pi n \sigma}}.$$

By the Markov property, with $k = \lfloor n/2 \rfloor$,

$$\begin{aligned}
&\mathbb{P}(y + S_n = z\,,\, \tau_y > n) \\
&= \int_0^{+\infty} \mathbb{P}(y' + S_k = z\,,\, \tau_{y'} > k)\, \mathbb{P}(y + S_{n-k} \in \mathrm{d}y'\,,\, \tau_y > n-k) \\
&\leqslant \frac{c}{\sqrt{k}} \mathbb{P}(\tau_y > n-k) \\
&\leqslant \frac{c}{\sqrt{k}\sqrt{n-k}} (1 + \max(0, y))\,.
\end{aligned}$$

Sketch of the proof of CLLT in i.i.d. case (1/3)

Corollary 4.5.5 (i.i.d.)

For any $n \geqslant 1$,

$$\sup_{y\in\mathbb{R}, z\geqslant 0} \mathbb{P}\left(y + S_n = z\right) \leqslant \frac{c}{\sqrt{n}}.$$

By the Markov property, with $k = \lfloor n/2 \rfloor$,

$$\begin{aligned}
&\mathbb{P}\left(y + S_n = z\,,\, \tau_y > n\right) \\
&\quad = \int_0^{+\infty} \mathbb{P}\left(y' + S_k = z\,,\, \tau_{y'} > k\right) \mathbb{P}\left(y + S_{n-k} \in \mathrm{d}y'\,,\, \tau_y > n-k\right) \\
&\quad \leqslant \frac{c}{\sqrt{k}} \mathbb{P}\left(\tau_y > n-k\right) \\
&\quad \leqslant \frac{c}{\sqrt{k}\sqrt{n-k}} \left(1 + \max(0, y)\right).
\end{aligned}$$

Sketch of the proof of CLLT in i.i.d. case (1/3)

Lemma 4.6.1 (i.i.d.)

For any $y \in \mathbb{R}$ and $n \geqslant 1$,

$$\sup_{z \geqslant 0} \mathbb{P}\left(y + S_n = z\,,\, \tau_y > n\right) \leqslant \frac{c}{n}\left(1 + \max(0, y)\right).$$

By the Markov property, with $k = \lfloor n/2 \rfloor$,

$$\begin{aligned}
&\mathbb{P}\left(y + S_n = z\,,\, \tau_y > n\right) \\
&= \int_0^{+\infty} \mathbb{P}\left(y' + S_k = z\,,\, \tau_{y'} > k\right) \mathbb{P}\left(y + S_{n-k} \in \mathrm{d}y'\,,\, \tau_y > n-k\right) \\
&\leqslant \frac{c}{\sqrt{k}} \mathbb{P}\left(\tau_y > n-k\right) \\
&\leqslant \frac{c}{\sqrt{k}\sqrt{n-k}}\left(1 + \max(0, y)\right).
\end{aligned}$$

Sketch of the proof of CLLT in i.i.d. case (1/3)

Lemma 4.6.1 (i.i.d.)

For any $y \in \mathbb{R}$ and $n \geqslant 1$,

$$\sup_{z \geqslant 0} \mathbb{P}\left(y + S_n = z\,,\ \tau_y > n\right) \leqslant \frac{c}{n}\left(1 + \max(0, y)\right).$$

$$X_1 \leftrightarrow X_n^*, \quad X_2 \leftrightarrow X_{n-1}^*, \quad \dots, \quad X_n \leftrightarrow X_1^*, \quad S_n \leftrightarrow -S_n^*.$$

Lemma 4.3.2 (duality, i.i.d.)

$$\begin{aligned}\mathbb{P}\left(y + S_n = z\,,\ \tau_y > n\right) &= \mathbb{P}\left(z + S_n^* = y\,,\ \tau_z^* > n\right)\\ &\leqslant \frac{c}{n}\left(1 + z\right).\end{aligned}$$

Sketch of the proof of CLLT in i.i.d. case (2/3)

Lemma 4.3.2 (duality, i.i.d.)

$$\mathbb{P}\left(y + S_k = z\,,\, \tau_{y'} > k\right) = \mathbb{P}\left(z + S_k^* = y\,,\, \tau_z^* > k\right) \leqslant \frac{c}{k}\left(1+z\right).$$

By the Markov property, with $k = \lfloor n/2 \rfloor$,

$$\begin{aligned}
&\mathbb{P}\left(y + S_n = z\,,\, \tau_y > n\right) \\
&= \int_0^{+\infty} \mathbb{P}\left(y' + S_k = z\,,\, \tau_{y'} > k\right) \mathbb{P}\left(y + S_{n-k} \in \mathrm{d}y'\,,\, \tau_y > n-k\right) \\
&\leqslant \frac{c}{k}\left(1+z\right) \mathbb{P}\left(\tau_y > n-k\right) \\
&\leqslant \frac{c}{k\sqrt{n-k}}\left(1+z\right)\left(1+\max(0,y)\right).
\end{aligned}$$

Sketch of the proof of CLLT in i.i.d. case (3/3)

D. Denisov, V. Wachtel (2015).
Random walks in cones. *The Annals of Probability*.

Lemma 4.6.2 (i.i.d.)

For any $y \in \mathbb{R}$, $z \geqslant 0$ and $n \geqslant 1$,

$$\mathbb{P}\left(y + S_n = z\,,\ \tau_y > n\right) \leqslant \frac{c}{n^{3/2}} \left(1 + \max(0, y)\right) \left(1 + z\right).$$

Sketch of the proof of CLLT in i.i.d. case (3/3)

D. Denisov, V. Wachtel (2015).
Random walks in cones. *The Annals of Probability*.

Lemma 4.6.2 (i.i.d.)

For any $y \in \mathbb{R}$, $z \geqslant 0$ and $n \geqslant 1$,

$$\mathbb{P}\left(y + S_n = z \,,\, \tau_y > n\right) \leqslant \frac{c}{n^{3/2}} \left(1 + \max(0, y)\right) (1 + z)\,.$$

→ Duality lemma for Markov chain.

→ The non-lattice case.

Hypotheses

- Let $(X_n)_{n \in \mathbb{N}}$ be a Markov chain on a finite state space $\mathbb{X}$.
- For a fixed function $f : \mathbb{X} \to \mathbb{R}$ and any $y \in \mathbb{R}$, the Markov walk is defined by

$$y + S_n = y + f(X_1) + \cdots + f(X_n), \qquad n \geqslant 1.$$

- The transition operator $\mathbf{P}$ of the Markov chain $(X_n)_{n \in \mathbb{N}}$ is assumed primitive.
- The function f is centred, $\boldsymbol{\nu}(f) = 0$, where $\boldsymbol{\nu}$ is the invariant measure.
- The Markov walk is non-lattice : for any $(a, \theta) \in \mathbb{R}^2$, there exists an orbit $x_0, \ldots, x_n$ in $\mathbb{X}$ such that

$$\mathbf{P}(x_0, x_1) \ldots \mathbf{P}(x_{n-1}, x_n)\mathbf{P}(x_n, x_0) > 0$$

and

$$f(x_0) + \cdots + f(x_n) - (n+1)\theta \notin a\mathbb{Z}.$$

Conditioned local limit theorems

→ Duality lemma for Markov chain.

The dual Markov transfer operator

For any $(x, x') \in \mathbb{X}^2$,

$$\mathbf{P}^*(x, x') := \frac{\boldsymbol{\nu}(x')}{\boldsymbol{\nu}(x)} \mathbf{P}(x', x).$$

→ The non-lattice case.

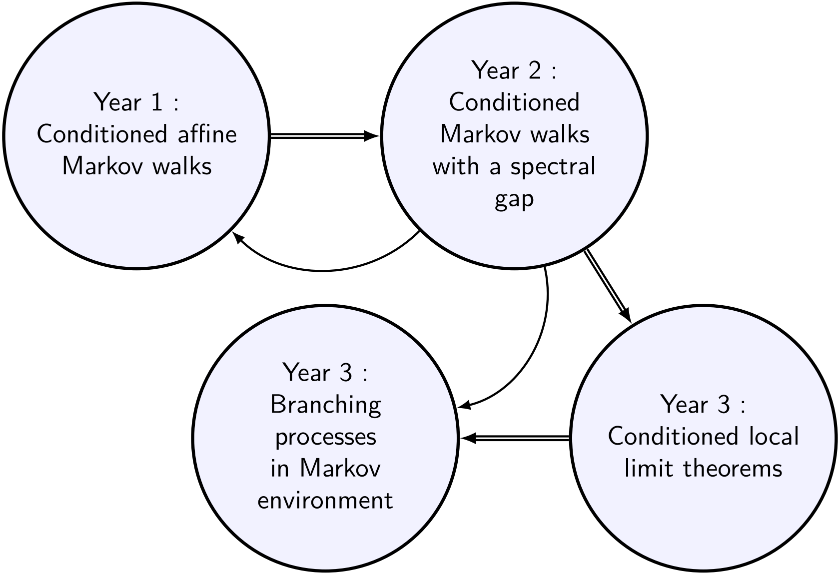
$$\mathbb{P}_x \left(y + S_n \in [z, z + a] \, , \, \tau_y > n \right) = \sum_{k=0}^{p-1} \mathbb{P}_x \left(y + S_n \in \left[z + \frac{ka}{p}, z + \frac{(k+1)a}{p} \right] \, , \, \tau_y > n \right).$$

Conditioned local limit theorems

Theorem 4.2.5

For any non-negative function $\psi : \mathbb{X} \to \mathbb{R}_+$, any $a > 0$, $x \in \mathbb{X}$, $y \in \mathbb{R}$ and $z \geqslant 0$,

$$\lim_{n \to +\infty} n^{3/2} \mathbb{E}_x \left(\psi(X_n) \, ; \, y + S_n \in [z, z+a] \, , \, \tau_y > n \right) = \frac{2V(x,y)}{\sqrt{2\pi}\sigma^3} \int_z^{z+a} \mathbb{E}_{\boldsymbol{\nu}}^* \left(\psi\left(X_1^*\right) V^*\left(X_1^*, z' + S_1^*\right) \, ; \, \tau_{z'}^* > 1 \right) \mathrm{d}z'.$$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The model

- Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain in a finite state space $\mathbb{X}$.
  $\rightarrow$ *X_n is the environment at the time n.*

- For any $i \in \mathbb{X}$, we consider a family $(\xi_i^{n,j})_{n,j\geqslant 1}$ of i.i.d. random variables. The branching process $(Z_n)_{n\geqslant 0}$ is defined recursively by $Z_0 = 1$ and for any $n \geqslant 1$,

$$Z_n = \sum_{j=1}^{Z_{n-1}} \xi_{X_n}^{n,j}$$

- For any $i \in \mathbb{X}$, let f_i be the common generated function of $\left(\xi_i^{n,j}\right)_{n,j\geqslant 1}$:

$$f_i(s) = \mathbb{E}\left(s^{\xi_i^{1,1}}\right), \qquad \text{for any } s \in [0,1].$$

Recall of the results in the independent case (1/2)

The critical case (Geiger, Kersting, 2001)

Suppose integrability assumptions and $\mathbb{E}\left(\ln\left(f'(1)\right)\right) = 0$. Then,

$$\mathbb{P}\left(Z_n > 0\right) \underset{n\to+\infty}{\sim} \frac{c}{\sqrt{n}}.$$

The strongly subcritical case (Guivarc'h, Liu, 2001)

Suppose integrability assumptions and $\mathbb{E}\left(f'(1)\ln\left(f'(1)\right)\right) < 0$. Then,

$$\mathbb{P}\left(Z_n > 0\right) \underset{n\to+\infty}{\sim} c\left[\mathbb{E}\left(f'(1)\right)\right]^n.$$

Recall of the results in the independent case (2/2)

The intermediate subcritical case (Geiger, Kersting, Vatutin, 2003)

Suppose integrability assumptions, $\mathbb{E}\left(\ln\left(f'(1)\right)\right) < 0$ and $\mathbb{E}\left(f'(1)\ln\left(f'(1)\right)\right) = 0$. Then,

$$\mathbb{P}\left(Z_n > 0\right) \underset{n\to+\infty}{\sim} \frac{c\left[\mathbb{E}\left(f'(1)\right)\right]^n}{\sqrt{n}}.$$

The weakly subcritical case (Geiger, Kersting, Vatutin, 2003)

Suppose integrability assumptions, $\mathbb{E}\left(\ln\left(f'(1)\right)\right) < 0$ and $\mathbb{E}\left(f'(1)\ln\left(f'(1)\right)\right) > 0$. Then, there exists $\gamma \in (0,1)$ such that

$$\mathbb{P}\left(Z_n > 0\right) \underset{n\to+\infty}{\sim} \frac{c\gamma^n}{n^{3/2}}.$$

The link with the Markov walk

A. Agresti (1974). Bound on the extinction time distribution of a branching process. *Advances in Applied Probability.*

Lemma 5.3.2

For any $i \in \mathbb{X}$ and $n \geqslant 1$,

$$\mathbb{P}_i\left(Z_n > 0\right) = \mathbb{E}_i\left(q_n\right) := \mathbb{E}_i\left(\left[e^{-S_n} + \sum_{k=0}^{n-1} e^{-S_k}\,\eta_{k+1,n}\right]^{-1}\right),$$

where $\left(\eta_{k+1,n}\right)_{n>k\geqslant 0}$ are uniformly bounded random variables,

$$S_n = \ln\left(f'_{X_1}(1)\cdots f'_{X_n}(1)\right) = \sum_{k=1}^{n} \rho\left(X_k\right)$$

and

$$\rho(i) = \ln\left(f'_i(1)\right), \qquad \text{for any } i \in \mathbb{X}.$$

The critical case in Markov environment : $\boldsymbol{\nu}(\rho) = 0$

The critical case

Assume that

$$\boldsymbol{\nu}\,(\rho) = \sum_{i \in \mathbb{X}} \ln\left(f_i'(1)\right) \boldsymbol{\nu}(i) = 0,$$

where $\boldsymbol{\nu}$ is the invariant measure of the Markov chain $(X_n)_{n \geqslant 1}$.

The critical case in Markov environment : $\nu(\rho) = 0$

The link with the Markov walk

The positive trajectories

The convergence of $\mathbb{E}^+_{i,y}(q_m)$

The conditional local limit theorem

The survival probability

The positive trajectories

$$q_n^{-1} = e^{-S_n} + \sum_{k=0}^{n-1} e^{-S_k} \eta_{k+1,n}.$$

Lemma 5.4.1

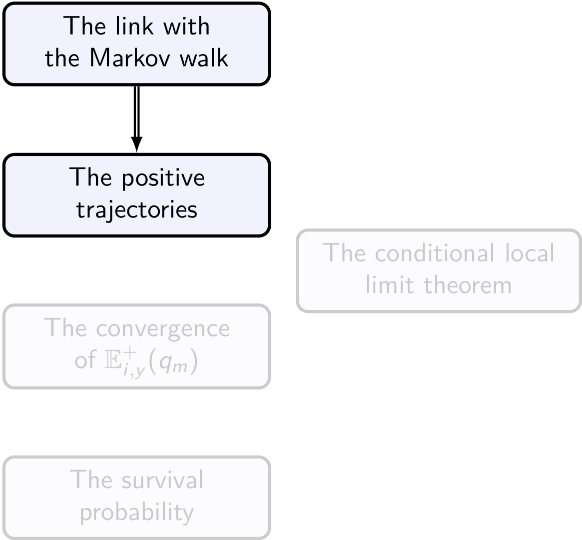
For any $m \geqslant 1$ and $(i, y) \in \operatorname{supp}(V)$,

$$\lim_{n \to +\infty} \mathbb{P}_i \left(Z_m > 0 \,|\, \tau_y > n \right) = \mathbb{E}_{i,y}^{+}(q_m),$$

where for any $k \geqslant 1$ and $g : \mathbb{X}^k \to \mathbb{C}$,

$$\mathbb{E}_{i,y}^{+}\left(g(X_1, \ldots, X_k) \right) = \frac{1}{V(i,y)} \mathbb{E}_i \left(g(X_1, \ldots, X_k) V(X_k, y + S_k) \,;\, \tau_y > k \right).$$

The critical case in Markov environment : $\nu(\rho) = 0$



The convergence of the process $\mathbb{E}^{+}_{i,y}\left(q_n\right)$

$$q_n^{-1} = \mathrm{e}^{-S_n} + \sum_{k=0}^{n-1} \mathrm{e}^{-S_k}\, \eta_{k+1,n}.$$

Lemma 5.3.13

For any $(i, y) \in \operatorname{supp}(V)$ and $k \geqslant 1$,

$$\mathbb{E}^{+}_{i,y}\left(\mathrm{e}^{-S_k}\right) \leqslant \frac{c\,(1+\max(0,y))\,\mathrm{e}^{y}}{k^{3/2} V(i,y)},$$

where we recall that $\mathbb{P}^{+}_{i,y}$ is the probability under which the trajectories $(y+S_n)_{n\geqslant 1}$ stay positive.

Lemma 5.4.3

For any $(i, y) \in \operatorname{supp}(V)$,

$$\lim_{m\to+\infty}\lim_{n\to+\infty} \mathbb{P}_i\left(Z_m > 0 \,|\, \tau_y > n\right) = \lim_{m\to+\infty} \mathbb{E}^{+}_{i,y}(q_m) = U(i,y).$$

The convergence of the process $\mathbb{E}^+_{i,y}\left(q_n\right)$

$$q_n^{-1} = \mathrm{e}^{-S_n} + \sum_{k=0}^{n-1} \mathrm{e}^{-S_k}\, \eta_{k+1,n}.$$

Lemma 5.3.13

For any $(i, y) \in \operatorname{supp}(V)$ and $k \geqslant 1$,

$$\mathbb{E}^+_{i,y}\left(\mathrm{e}^{-S_k}\right) \leqslant \frac{c\,(1+\max(0,y))\,\mathrm{e}^{y}}{k^{3/2}V(i,y)},$$

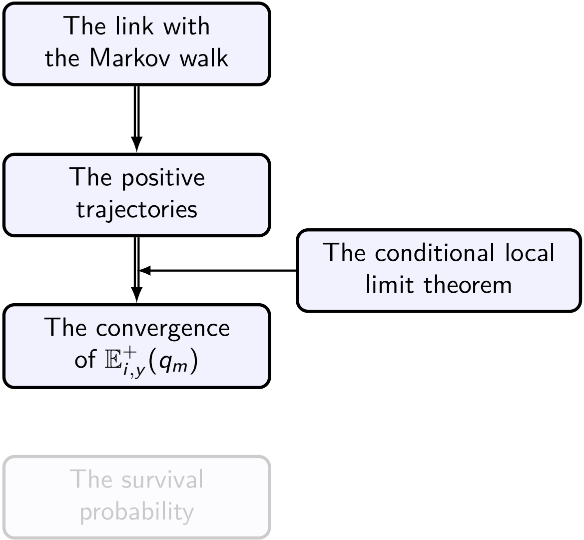
where we recall that $\mathbb{P}^+_{i,y}$ is the probability under which the trajectories $(y+S_n)_{n\geqslant 1}$ stay positive.

Lemma 5.4.6

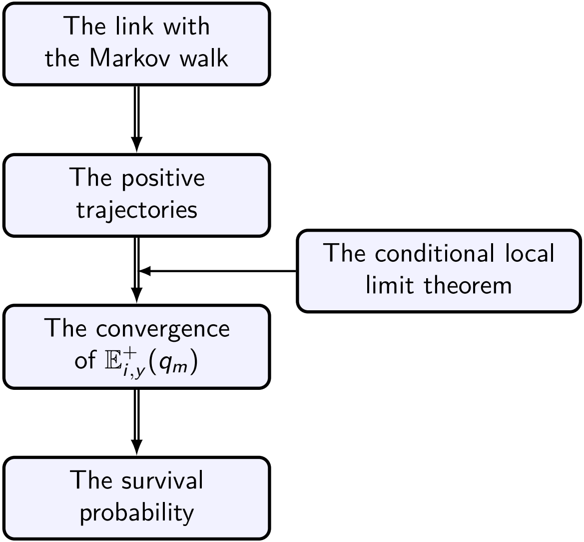
For any $(i, y) \in \operatorname{supp}(V)$,

$$\lim_{n\to+\infty} \mathbb{P}_i\left(Z_n > 0 \,|\, \tau_y > n\right) = U(i,y).$$

The critical case in Markov environment : $\nu(\rho) = 0$



The critical case in Markov environment : $\nu(\rho) = 0$



Result in the critical case in Markov environment

Theorem 5.2.1

There exists a positive function u on $\mathbb{X}$ such that, for any $(i,j) \in \mathbb{X}^2$,

$$\mathbb{P}_i\left(Z_n > 0\,,\; X_n = j\right) \underset{n\to+\infty}{\sim} \frac{\boldsymbol{\nu}(j)u(i)}{\sqrt{n}}.$$

The subcritical case in Markov environment : $\nu(\rho) < 0$

For any $\lambda \in \mathbb{R}$, consider $\mathbf{P}_\lambda$ defined by

$$\mathbf{P}_\lambda g(i) = \mathbf{P}\left(\mathrm{e}^{\lambda\rho} g\right)(i) = \mathbb{E}_i\left(\mathrm{e}^{\lambda S_1} g\left(X_1\right)\right),$$

for any $i \in \mathbb{X}$ and any function $g : \mathbb{X} \to \mathbb{C}$.

The subcritical case in Markov environment : $\boldsymbol{\nu}(\rho) < 0$

For any $\lambda \in \mathbb{R}$, consider $\mathbf{P}_\lambda$ defined by

$$\mathbf{P}_\lambda g(i) = \mathbf{P}\left(\mathrm{e}^{\lambda\rho} g\right)(i) = \mathbb{E}_i\left(\mathrm{e}^{\lambda S_1} g\left(X_1\right)\right),$$

for any $i \in \mathbb{X}$ and any function $g : \mathbb{X} \to \mathbb{C}$.The operator $\mathbf{P}_\lambda$ has the following decomposition :

$$\mathbf{P}_\lambda g(i) = k(\lambda)\boldsymbol{\nu}_\lambda(g)v_\lambda(i) + Q_\lambda(g)(i),$$

where

- $k(\lambda) > 0$ is an eigenvalue of $\mathbf{P}_\lambda$ and its spectral radius,
- $\boldsymbol{\nu}_\lambda$ is a positive linear form,
- v_λ is a positive function on $\mathbb{X}$ and an eigenvector of $\mathbf{P}_\lambda$,
- Q_λ is an operator with a spectral radius strictly less than $k(\lambda)$.

The subcritical case in Markov environment : $\boldsymbol{\nu}(\rho) < 0$

For any $\lambda \in \mathbb{R}$, consider $\mathbf{P}_\lambda$ defined by

$$\mathbf{P}_\lambda g(i) = \mathbf{P}\left(e^{\lambda\rho} g\right)(i) = \mathbb{E}_i\left(e^{\lambda S_1} g\left(X_1\right)\right),$$

for any $i \in \mathbb{X}$ and any function $g : \mathbb{X} \to \mathbb{C}$.The operator $\mathbf{P}_\lambda$ has the following decomposition :

$$\mathbf{P}_\lambda g(i) = k(\lambda)\boldsymbol{\nu}_\lambda(g)v_\lambda(i) + Q_\lambda(g)(i),$$

where

- $k(\lambda) > 0$ is an eigenvalue of $\mathbf{P}_\lambda$ and its spectral radius,
- $\boldsymbol{\nu}_\lambda$ is a positive linear form,
- v_λ is a positive function on $\mathbb{X}$ and an eigenvector of $\mathbf{P}_\lambda$,
- Q_λ is an operator with a spectral radius strictly less than $k(\lambda)$.

The operator

$$\tilde{\mathbf{P}}_\lambda g(i) = \frac{\mathbf{P}_\lambda\left(g v_\lambda\right)(i)}{k(\lambda)v_\lambda(i)} = \frac{\mathbb{E}_i\left(e^{\lambda S_1} g\left(X_1\right) v_\lambda\left(X_1\right)\right)}{k(\lambda)v_\lambda(i)},$$

is a Markov operator.

The subcritical case in Markov environment : $\boldsymbol{\nu}(\rho) < 0$

The drift of the walk under the changed measure is given by

$$\tilde{\boldsymbol{\nu}}_\lambda(\rho) = \boldsymbol{\nu}_\lambda\left(\rho v_\lambda\right) = \sum_{i \in \mathbb{X}} \ln\left(f_i'(1)\right) v_\lambda(i) \boldsymbol{\nu}_\lambda(i).$$

Lemma 5.3.15

The function $K : \lambda \to \ln(k(\lambda))$ is strictly convex and satisfies :

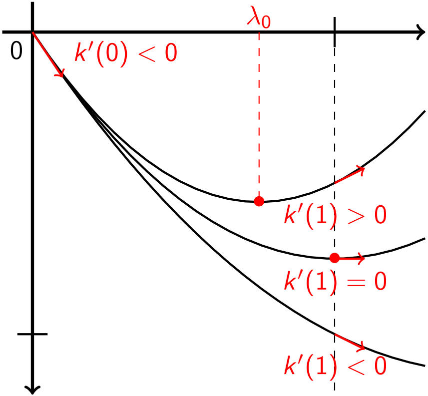
$$K'(\lambda) = \frac{k'(\lambda)}{k(\lambda)} = \tilde{\boldsymbol{\nu}}_\lambda(\rho)$$

and

$$K''(\lambda) = \tilde{\boldsymbol{\nu}}_\lambda\left(\rho^2\right) - \tilde{\boldsymbol{\nu}}_\lambda(\rho)^2 + 2\sum_{n=1}^{+\infty}\left[\tilde{\boldsymbol{\nu}}_\lambda\left(\rho \tilde{\mathbf{P}}_\lambda^n \rho\right) - \tilde{\boldsymbol{\nu}}_\lambda(\rho)^2\right] > 0.$$

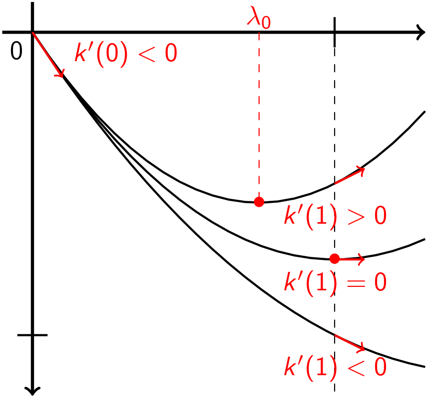
Y. Guivarc'h and J. Hardy (1988). Théorèmes limites pour une classe de chaîne de Markov et applications aux difféomorphismes d'Anosov. *Annales de l'IHP Probabilités et statistiques*.

The subcritical case in Markov environment : $\nu(\rho) < 0$



$$\mathbb{E}_i(q_n) = k(\lambda)^n v_\lambda(i) \tilde{\mathbb{E}}_i \left(\frac{e^{-\lambda S_n}}{v_\lambda(X_n)} \left[e^{-S_n} + \sum_{k=0}^{n-1} e^{-S_k} \eta_{k+1,n} \right]^{-1} \right).$$

The subcritical case in Markov environment : $\nu(\rho) < 0$



Taking $\lambda = 1$, we obtain

$$\mathbb{E}_i\left(q_n\right) = k(1)^n v_1(i)\tilde{\mathbb{E}}_i\left(\frac{1}{v_1\left(X_n\right)}\left[1 + \sum_{k=0}^{n-1} e^{S_n - S_k}\,\eta_{k+1,n}\right]^{-1}\right).$$

Results in the subcritical case in Markov environment

Theorem 5.2.2 (strongly subcritical case)

If $k'(0) < 0$ and $k'(1) < 0$ then there exists a positive function u on $\mathbb{X}$ such that for any $(i,j) \in \mathbb{X}^2$,

$$\lim_{n\to+\infty} \mathbb{P}_i\left(Z_n > 0\,,\, X_n = j\right) \underset{n\to+\infty}{\sim} k(1)^n v_1(i) u(j).$$

Theorem 5.2.3 (intermediate subcritical case)

If $k'(0) < 0$ and $k'(1) = 0$ then there exists a positive function u on $\mathbb{X}$ such that for any $(i,j) \in \mathbb{X}^2$,

$$\lim_{n\to+\infty} \mathbb{P}_i\left(Z_n > 0\,,\, X_n = j\right) \underset{n\to+\infty}{\sim} \frac{k(1)^n}{\sqrt{n}} v_1(i) u(j).$$

Results in the subcritical case in Markov environment

Theorem 5.2.3 (weakly subcritical case)

If $k'(0) < 0$ and $k'(1) > 0$ then there exist a unique $\lambda_0 \in (0,1)$ and a positive function u on $\mathbb{X}^2$ such that for any $(i,j) \in \mathbb{X}^2$,

$$\lim_{n \to +\infty} \mathbb{P}_i \left(Z_n > 0 \,,\, X_n = j \right) \underset{n \to +\infty}{\sim} \frac{k(\lambda_0)^n}{n^{3/2}} u(i,j).$$

Publications

- Limit theorems for affine Markov walks conditioned to stay positive.
  Annales de l'institut Henri Poincaré, (B) Probabilités et Statistiques, 2016 (in press).

- Limit theorems for Markov walks conditioned to stay positive under a spectral gap assumption.
  Annals of Probability, 2017 (in press).

- Conditioned local limit theorems for Markov walks defined on finite Markov chains, 2017 (preprint).

- The survival probability of critical and subcritical branching processes in finite state space Markovian environment, 2017 (preprint).